# 12.4) The Cross Product

**1. The Cross Product Definition and Basic Properties**:

There are three types of multiplication involving vectors. First, we have *scalar multiplication of a vector*, which we learned about in Section 12.2. The remaining two types involve multiplying one vector by another vector. The first of these is the *dot product*, which we learned about in Section 12.3. In this section, we will study the other method for multiplying a vector by a vector, which is known as the *cross product*. (At the end of this section, we will also learn about the *box product*, which is a combination of the cross product and the dot product.)

Whereas the *dot* product of two vectors produces a *scalar*, the *cross* product of two vectors produces a *vector*. For this reason, the dot product is also known as the **scalar product** and the cross product is also known as the **vector product**.

The cross product is defined *only* for three-dimensional vectors. In contrast, the dot product can be performed on either two-dimensional or three-dimensional vectors. Thus, in the following discussion, all vectors are assumed to be three-dimensional unless otherwise specified.

Given any two vectors $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{a} \times \mathbf{b}$, and is defined by the formula $< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$.

One way to remember the cross product is through the use of determinants...

Given $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, $\mathbf{a} \times \mathbf{b} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

By the Cofactor Expansion along the first row, we get

$$\det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\mathbf{i} - \det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\mathbf{j} + \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\mathbf{k}$$

Alternative notation:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Another way to remember the cross product is through the use of the "Shea Triangle." Draw an equilateral triangle. Label the lower left vertex $1$, the middle top vertex $2$, and the lower right vertex $3$. Start at the top and move full-circle around the triangle in the clockwise direction. Use each consecutive pair of numbers to fill in the subscripts for $a_m b_n - a_n b_m$. This wil give you the first, second, and third components of $\mathbf{a} \times \mathbf{b}$.

**Example One**: If $\mathbf{a} = < 1,3,4 >$ and $\mathbf{b} = < 2,7,-5 >$, then $\mathbf{a} \times \mathbf{b} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k} =$$

$(-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} = < -43, 13, 1 >$.

In the above example, note that $< -43, 13, 1 > \cdot < 1,3,4 > = 0$ and $< -43, 13, 1 > \cdot < 2,7,-5 > = 0$, in other words, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Is this a coincidence? The answer is no, as the following theorem makes clear...

**Theorem 1**: $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

We shall prove that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$. (The proof of its orthogonality to $\mathbf{b}$ is similar.)
$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 > \cdot < a_1, a_2, a_3 > =$
$(a_2b_3 - a_3b_2)a_1 + (a_3b_1 - a_1b_3)a_2 + (a_1b_2 - a_2b_1)a_3 =$
$a_1a_2b_3 - a_1b_2a_3 + b_1a_2a_3 - a_1a_2b_3 + a_1b_2a_3 - b_1a_2a_3 = 0$.

**Corollary 1 to Theorem 1**: $\mathbf{a} \times \mathbf{b}$ is orthogonal to any scalar multiple of $\mathbf{a}$ or $\mathbf{b}$.

We shall prove that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $k\mathbf{a}$. (The proof of its orthogonality to $k\mathbf{b}$ is similar.)
$(\mathbf{a} \times \mathbf{b}) \cdot (k\mathbf{a}) = k(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = k(0) = 0$.

**Example Two**: As in Example One, let $\mathbf{a} = < 1,3,4 >$ and $\mathbf{b} = < 2,7,-5 >$, so $\mathbf{a} \times \mathbf{b} = < -43, 13, 1 >$. $5\mathbf{a} = < 5, 15, 20 >$ and $-3\mathbf{b} = < -6, -21, 15 >$. By Corollary 1 to Theorem 1, $< -43, 13, 1 >$ is orthogonal to both $< 5, 15, 20 >$ and $< -6, -21, 15 >$.

**Corollary 2 to Theorem 1**: $\mathbf{a} \times \mathbf{b}$ is orthogonal to any linear combination of $\mathbf{a}$ and $\mathbf{b}$. In other words, if $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$.

This follows from Corollary 1: Let $\mathbf{c} = p\mathbf{a} + q\mathbf{b}$. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} =$
$(\mathbf{a} \times \mathbf{b}) \cdot (p\mathbf{a} + q\mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot (p\mathbf{a}) + (\mathbf{a} \times \mathbf{b}) \cdot (q\mathbf{b}) = 0 + 0 = 0$.

**Example Three**: As in Example One, let $\mathbf{a} = < 1,3,4 >$ and $\mathbf{b} = < 2,7,-5 >$, so $\mathbf{a} \times \mathbf{b} = < -43, 13, 1 >$. $7\mathbf{a} + 2\mathbf{b} = < 11, 35, 18 >$. By Corollary 2, $< -43, 13, 1 >$ is orthogonal to $< 11, 35, 18 >$.

**Example Four**: Let $\mathbf{a} = < 4,1,3 >$ and $\mathbf{b} = < 3,4,5 >$. $\mathbf{a} \times \mathbf{b} = < -7, -11, 13 >$. Let $\mathbf{c} = 3\mathbf{a} - 2\mathbf{b} = < 6, -5, -1 >$. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ by Theorem 1 and to $\mathbf{c}$ by Corollary 2. As an exercise, you should confirm this by computing the dot product of $< -7, -11, 13 >$ with $< 4,1,3 >$, $< 3,4,5 >$, and $< 6, -5, -1 >$.

Actually, Corollary 1 is a special case of Corollary 2, since any scalar multiple of $\mathbf{a}$ or $\mathbf{b}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$ ($k\mathbf{a} = k\mathbf{a} + 0\mathbf{b}$, and $k\mathbf{b} = 0\mathbf{a} + k\mathbf{b}$). Furthermore, Theorem 1 itself is a special case of both Corollary 1 and Corollary 2, since $\mathbf{a} = 1\mathbf{a} = 1\mathbf{a} + 0\mathbf{b}$ and $\mathbf{b} = 1\mathbf{b} = 0\mathbf{a} + 1\mathbf{b}$.

It should come as no surprise that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, since the cross product was basically *invented* for the purpose of finding a vector orthogonal to any given pair of three-dimensional vectors. In other words, mathematicians *started* with the goal of coming up with an operation that would give them this result, and then they *figured out* a formula that would accomplish this purpose. It is illuminating to consider the thought process by which they discovered the formula for the cross product...

Say we have the vectors $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, and we wish to find a vector $\mathbf{c} = < c_1, c_2, c_3 >$ so that $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$. Obviously, $\mathbf{c} = \mathbf{0}$ would work, but this is trivial; can we find a nontrivial answer?

$\mathbf{a} \cdot \mathbf{c} = a_1c_1 + a_2c_2 + a_3c_3$, and $\mathbf{b} \cdot \mathbf{c} = b_1c_1 + b_2c_2 + b_3c_3$, so we must find a solution to the system of linear equations:

**1**. $a_1c_1 + a_2c_2 + a_3c_3 = 0$
**2**. $b_1c_1 + b_2c_2 + b_3c_3 = 0$

We multiply equation 1 by $b_3$ and equation 2 by $-a_3$, giving us

**1**. $a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 = 0$
**2**. $-a_3b_1c_1 - a_3b_2c_2 - a_3b_3c_3 = 0$

Adding these two equations gives us
$a_1b_3c_1 - a_3b_1c_1 + a_2b_3c_2 - a_3b_2c_2 = 0$, which factors as
$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$, so
$(a_2b_3 - a_3b_2)c_2 = -(a_1b_3 - a_3b_1)c_1$, which we may rewrite as
$(a_2b_3 - a_3b_2)c_2 = (a_3b_1 - a_1b_3)c_1$.
We obtain a solution if we choose $c_1 = a_2b_3 - a_3b_2$ and $c_2 = a_3b_1 - a_1b_3$.
If we substitute these into our original equation 1, we get:
$a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3c_3 = 0$
Distributing gives us
$a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3c_3 = 0$
Canceling opposite terms gives us
$-a_1a_3b_2 + a_2a_3b_1 + a_3c_3 = 0$
$a_3c_3 = a_1a_3b_2 - a_2a_3b_1$
$a_3c_3 = a_3(a_1b_2 - a_2b_1)$
We obtain a solution if we choose $c_3 = a_1b_2 - a_2b_1$.
Thus, we obtain $\mathbf{c} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$.

By the way, the reason the cross product was defined *only* for three-dimensional vectors is that in two-dimensional space, it is *impossible* to find a nonzero vector orthogonal to two given nonzero vectors (unless the two vectors are parallel). For example, consider $\mathbf{a} = < 1, 1 >$ and $\mathbf{b} = < 1, -1 >$. Suppose we could find a nonzero vector $\mathbf{c} = < c_1, c_2 >$ such that $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$. So:

1. $c_1 + c_2 = 0$
2. $c_1 - c_2 = 0$

Adding the two equations gives us $2c_1 = 0$, so $c_1 = 0$. This implies $c_2 = 0$, contradicting our assumption that $\mathbf{c}$ is nonzero.

**Theorem 2**: For any vector $\mathbf{a}$, $\mathbf{a} \times \mathbf{0} = \mathbf{0}$, $\mathbf{0} \times \mathbf{a} = \mathbf{0}$, and $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

The first two equations are true because, in the formula for the cross product, if every term has a factor of $0$, then we get $0$ for every term, so we get $0$ for every component. For the third equation, $\mathbf{a} \times \mathbf{a} = < a_2a_3 - a_3a_2, a_3a_1 - a_1a_3, a_1a_2 - a_2a_1 > = < 0,0,0 >$.

**Example Five**: $< 18,-7,29 > \times < 18,-7,29 > = \mathbf{0}$, by Theorem 2.

**Theorem 3**: If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors and $\theta$ is the angle between them, then $|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$.

The proof of this theorem is presented on page 817 of the text. Here is a brief summary: First we show that $|\mathbf{a} \times \mathbf{b}|^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2$. We know from the Dot Product Theorem that $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$. Hence, substitution gives us $a^2b^2 - a^2b^2\cos^2\theta$, or $a^2b^2(1 - \cos^2\theta)$, which equals $a^2b^2\sin^2\theta$. Since $\theta \in [0,\pi]$, $\sin\theta \geq 0$, so taking square roots gives us $|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$.

**Corollary 1 to Theorem 3**: Two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$. (Thus, two nonzero, non-parallel vectors must have a nonzero cross product.)

This follows directly from Theorem 3, since $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\sin\theta = 0$.

**Corollary 2 to Theorem 3**: Two vectors have a cross product of $\mathbf{0}$ if and only if one vector is a scalar multiple of the other.

Corollary 2 is a slight broadening of Corollary 1; we drop the assumption that the vectors are nonzero, but then we must describe them as having a scalar multiple relationship (we cannot refer to them as parallel if one might be zero). (Bear in mind that $\mathbf{0}$ is a scalar multiple of every vector–it is $0$ times the vector.)

**Example Six**: Let $\mathbf{a} = < 4,-3,-2 >$ and $\mathbf{b} = < -12,9,6 >$. $\mathbf{b} = -3\mathbf{a}$, so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Interestingly, Theorem 2 follows from Corollary 2 to Theorem 3. For any vector $\mathbf{a}$, $\mathbf{0} = 0\mathbf{a}$ and $\mathbf{a} = 1\mathbf{a}$. Thus:

- $\mathbf{a} \times \mathbf{0} = \mathbf{a} \times 0\mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = 0\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{a} = \mathbf{a} \times 1\mathbf{a} = \mathbf{0}$

**Theorem 4**: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero, non-parallel vectors. If $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$, then $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

**Proof**:

Let $\mathbf{a} = < a_1, a_2, a_3 >$, $\mathbf{b} = < b_1, b_2, b_3 >$, and $\mathbf{c} = < c_1, c_2, c_3 >$. Assume $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel, and $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel, $\mathbf{a} \times \mathbf{b}$ is nonzero. Hence, at least one of its three components must be nonzero. There are three cases to consider; we will address only one (the other two are analogous). Say the third component, $a_1b_2 - a_2b_1$, is nonzero.

We must show that there exist scalars $p$ and $q$ such that $\mathbf{c} = p\mathbf{a} + q\mathbf{b}$. Thus, $p$ and $q$ must satisfy the following system of linear equations:

**1**. $a_1p + b_1q = c_1$
**2**. $a_2p + b_2q = c_2$
**3**. $a_3p + b_3q = c_3$

We multiply equation 1 by $-a_2$ and equation 2 by $a_1$, giving us

**1**. $-a_1a_2p - a_2b_1q = -a_2c_1$
**2**. $a_1a_2p + a_1b_2q = a_1c_2$

Adding these two equations gives us
$a_1b_2q - a_2b_1q = a_1c_2 - a_2c_1$, which factors as
$(a_1b_2 - a_2b_1)q = a_1c_2 - a_2c_1$, so
$q = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Returning to our original three equations, we multiply equation 1 by $b_2$ and equation 2 by $-b_1$, giving us

**1**. $a_1b_2p + b_1b_2q = b_2c_1$
**2**. $-a_2b_1p - b_1b_2q = -b_1c_2$

Adding these two equations gives us
$a_1b_2p - a_2b_1p = b_2c_1 - b_1c_2$, which factors as
$(a_1b_2 - a_2b_1)p = b_2c_1 - b_1c_2$, so
$p = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$

To confirm that we have found a legitimate solution to the system, we must verify that these values of $p$ and $q$ satisfy equation 3. If we substitute these values into equation 3, we obtain the equation $a_3\left(\frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right) = c_3$. One of our premises is that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$, i.e., $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. It turns out that these two equations are equivalent (you may work out the details as an exercise).

Hence, we have found scalars $p$ and $q$ such that $\mathbf{c} = p\mathbf{a} + q\mathbf{b}$, i.e., we have shown that $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

**QED**.

Theorem 4 and Corollary 2 to Theorem 1 can be combined together into one theorem. However, since Theorem 4 requires $\mathbf{a}$ and $\mathbf{b}$ to be nonzero and non-parallel, this assumption must be applied to the combined result...

**Theorem 5**: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero, non-parallel vectors. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$ if and only if $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

## 2. Geometric Interpretation of the Cross Product:

In $x,y,z$ space, a **plane** is a set of points satisfying an equation of the form $ax + by + cz = d$, where $a$, $b$, $c$, and $d$ are constants and where $a$, $b$, and $c$ are not all zero. (Planes will be studied in great detail in Section 12.5.)

We say that a representation of a vector **lies in** a given plane if both the tail and the tip are in the plane. When this is the case, *every* point of the representation is in the plane.

We say that a vector **belongs to** a given plane if the vector has a representation that lies in the plane. When this is the case, any representation of the vector that intersects the plane must lie in the plane (if *any* point of the representation is in the plane, then *every* point of the representation is in the plane).

Of course, you should bear in mind that a vector is an equivalence class comprising infinitely many oriented line segments. If a vector belongs to a plane, it has infinitelty many representations that lie in the plane, but it also has infinitely many representations that do not lie in the plane.

To determine whether or not a given vector belongs to a given plane, we choose any point in the plane to serve as the tail, and then we examine whether the tip is also in the plane.

**Example Seven**: The vector $\mathbf{i} = < 1,0,0 >$ belongs to the plane $z = 5$, because if its tail is $(3,7,5)$, then its tip is $(4,7,5)$, and both of these points lie in the plane. On the other hand, $\mathbf{i}$ does *not* belong to the plane $x = 1$, because if its tail is $(1,6,9)$, which lies in the plane, then its tip is $(2,6,9)$, which does not lie in the plane.

If a vector belongs to a given plane, then it also belongs to every plane parallel to that plane. In Example Seven, for instance, $\mathbf{i}$ belongs not only to the plane $z = 5$, but also to every horizontal plane. (In particular, note that the standard-position representation of $\mathbf{i}$ has tail $(0,0,0)$ and tip $(1,0,0)$ and lies in the plane $z = 0$.)

Since $\mathbf{0} = < 0,0,0 >$ is represented by a single point, rather than by a directed line segment, $\mathbf{0}$ belongs to every plane. (Every point in every plane is a representation of the zero vector.)

A plane is *uniquely determined* by two nonzero, non-parallel vectors and a given point. In other words, given nonzero, non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ and a point $P$, there is a unique plane to which $\mathbf{a}$ and $\mathbf{b}$ both belong and which contains $P$. To visualize this plane, let $\mathbf{a}$ and $\mathbf{b}$ both be placed at $P$. Let $A$ be the tip of $\mathbf{a}$ and let $B$ be the tip of $\mathbf{b}$ (in other words, let $\overrightarrow{PA}$ represent $\mathbf{a}$ and $\overrightarrow{PB}$ represent $\mathbf{b}$). The points $A$, $B$, and $P$ determine a unique plane, which contains the triangle $\Delta PAB$. We name this plane $PAB$.

Now let $\mathbf{c}$ be any vector. Let $\mathbf{c}$ be placed at $P$, and let $C$ be its tip. $C$ may or may not be in $PAB$, so $\overrightarrow{PC}$ may or may not lie in $PAB$, so $\mathbf{c}$ may or may not belong to $PAB$. If $\mathbf{c}$ does belong to $PAB$, then $\mathbf{c}$ is said to be **coplanar with** $\mathbf{a}$ and $\mathbf{b}$.

More generally, any collection of vectors is said to be **coplanar** if they all belong to a common plane, and is said to be **non-coplanar** if they do *not* all belong to a common plane (i.e., if there is no plane to which all the vectors belong). To determine whether or not the vectors are coplanar, we choose any point in $x,y,z$ space to serve as a common tail for all the vectors, and then we examine whether that tail and all the tips lie in one plane. (If the common tail is the origin, then we are examining the standard-position representations of the vectors.)

Any two vectors are necessarily coplanar, because any three points lie in a common plane. Likewise, any two vectors and the zero vector are necessarily coplanar. Furthermore, given three nonzero vectors, if any two of them are parallel, then the three are necessarily coplanar. In order for the issue to be nontrivial, we must be dealing with at least *three* vectors that are *nonzero* and *pairwise non-parallel*. In other words, if we have three or more vectors that are nonzero and pairwise non-parallel, they may be either coplanar or non-coplanar.

Clarification: When we say that three vectors are *pairwise* non-parallel, we mean that no two of them are parallel. If we merely say that three vectors are non-parallel, this means it is not the case that all three are parallel. Pairwise non-parallel is a much stronger condition!

If we state that three vectors are non-coplanar, this implies they are nonzero and pairwise non-parallel. (Being nonzero and pairwise non-parallel is a necessary condition, but not a sufficient condition, for being non-coplanar.)

The following two examples illustrate how three nonzero, pairwise non-parallel vectors may be either coplanar or non-coplanar. In the first example, they are non-coplanar, but in the second example, they are coplanar.

**Example Eight**: The vectors $\mathbf{i} = < 1,0,0 >$, $\mathbf{j} = < 0,1,0 >$, and $\mathbf{k} = < 0,0,1 >$ are non-coplanar. If we examine their standard-position representations, they have tail $(0,0,0)$ and tips $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and these four points do not lie in one plane. (This fact should be geometrically obvious, but we can confirm it algebraically: For any plane $ax + by + cz = d$, if the four points satisfy the equation, we can infer $a = b = c = 0$, which is a contradiction.)

**Example Nine**: The vectors $\mathbf{i} = < 1,0,0 >$, $\mathbf{j} = < 0,1,0 >$, and $< 2,3,0 >$ are coplanar. If we examine their standard-position representations, they have tail $(0,0,0)$ and tips $(1,0,0)$, $(0,1,0)$, and $(2,3,0)$, and these four points lie in one plane, namely, the plane $z = 0$.

In Example Nine, notice that $< 2,3,0 > = 2\mathbf{i} + 3\mathbf{j}$, i.e., $< 2,3,0 >$ is a linear combination of $\mathbf{i}$ and $\mathbf{j}$. In fact, *any* linear combination of *any* two vectors is coplanar with those two vectors, as the following theorem makes clear...

**Theorem 6**: If $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, then the three vectors are coplanar.

The converse is true *provided* we assume that $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel. This fact, in conjunction with Theorem 6, gives us an "if and only if" relationship, as the following theorem makes clear...

**Theorem 7**: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero, non-parallel vectors. $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Theorems 5 and 7 can be combined into one theorem...

**Theorem 8**: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero, non-parallel vectors. The following three conditions are equivalent (i.e., each one implies the other two):

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$, i.e., $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$.
2. $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.
3. $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Actually, the equivalence of conditions 1 and 2 in the above listing does *not* depend on vectors $\mathbf{a}$ and $\mathbf{b}$ being nonzero and non-parallel. Thus, we can state their equivalence as a separate theorem...

**Theorem 9**: A vector is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if it is orthogonal to $\mathbf{a} \times \mathbf{b}$.

*Proof:* (1) If $\mathbf{c}$ is non-coplanar with $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel, so (by Theorem 8) $\mathbf{c}$ is not orthogonal to $\mathbf{a} \times \mathbf{b}$. (2) Assume $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}$ or $\mathbf{b}$ is zero, of if $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \times \mathbf{b}$ is zero, so $\mathbf{c}$ is orthogonal to $\mathbf{a} \times \mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel, then (by Theorem 8) $\mathbf{c}$ is orthogonal to $\mathbf{a} \times \mathbf{b}$.

**Example 10**: Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors defined in Example Four, namely, $\mathbf{a} = < 4,1,3 >$, $\mathbf{b} = < 3,4,5 >$, and $\mathbf{c} = < 6,-5,-1 >$. We saw in Example Four that $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, and is orthogonal to $\mathbf{a} \times \mathbf{b}$. We can now infer (from either Theorem 8 or 9) that $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$. To make this less abstract, let us examine a specific plane to which $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all belong. Let us choose $P$ to be the point $(5,2,6)$. Then $A = (9,3,9)$, $B = (8,6,11)$, and $C = (11,-3,5)$. These four points lie in one plane. As we shall see in Section 12.5, the equation of this plane is $7x + 11y - 13z = -21$. For now, you may confirm that each of the four points satisfies this equation. Alternatively, if we place the three vectors in standard position, then we will have $P = (0,0,0)$, $A = (4,1,3)$, $B = (3,4,5)$, and $C = (6,-5,-1)$, and these four points lie in the plane $7x + 11y - 13z = 0$. (Again, the details will be made clear in Section 12.5, but for now, just confirm the four points satisfy this equation.)

**Example 11**: Let $\mathbf{a} = < 1,2,0 >$ and $\mathbf{b} = < 5,4,0 >$. $\mathbf{a} \times \mathbf{b} = < 0,0,-6 >$. By Theorem 7 or 8, these three vectors are non-coplanar because $< 0,0,-6 >$ is not a linear combination of $< 1,2,0 >$ and $< 5,4,0 >$ (any linear combination of those two vectors would have to have $0$ as its third component).

We may generalize from Example 11: If $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel, then $\mathbf{a} \times \mathbf{b}$ is non-coplanar with $\mathbf{a}$ and $\mathbf{b}$. On the other hand, if either $\mathbf{a}$ or $\mathbf{b}$ is zero or the two are parallel, then $\mathbf{a} \times \mathbf{b}$ is zero and is therefore coplanar with $\mathbf{a}$ and $\mathbf{b}$. Thus we have the following theorem...

**Theorem 10**: $\mathbf{a} \times \mathbf{b}$ is non-coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel. Equivalently: $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if one of the vectors is zero or the two vectors are parallel. (Thus, for nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.)

*Proof:* By Theorem 9, $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a} \times \mathbf{b}$. The only vector orthogonal to itself is the zero vector. Hence, $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. By Corollary 2 to Theorem 3, the cross product of two vectors is zero if and only if one vector is a scalar multiple of the other–i.e., one of the vectors is zero or the two vectors are parallal. Ergo, $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if one of the vectors is zero or the two vectors are parallel.

We stated earlier that that two nonzero, non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ and a point $P$ determine a unique plane, which we named $PAB$ (given $\overrightarrow{PA}$ representing $\mathbf{a}$ and $\overrightarrow{PB}$ representing $\mathbf{b}$). $\mathbf{a}$ and $\mathbf{b}$ belong to $PAB$, but (by Theorem 10) $\mathbf{a} \times \mathbf{b}$ does not. Let $\mathbf{a} \times \mathbf{b}$ be placed at $P$, and let $R$ denote its tip (i.e., let $\overrightarrow{PR}$ represent $\mathbf{a} \times \mathbf{b}$). $R$ is not in $PAB$. Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, $\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PA}$ and $\overrightarrow{PB}$. Let $C$ be any point in $PAB$ distinct from $P$, and let $\mathbf{c}$ be the nonzero vector represented by $\overrightarrow{PC}$. $\overrightarrow{PC}$ lies in $PAB$, so $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$ (by Theorem 8 or 9), $\overrightarrow{PR}$ is perpendicular to $\overrightarrow{PC}$.

Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to *every* vector belonging to $PAB$, we may say it is orthogonal to the plane itself. In general, $\mathbf{a} \times \mathbf{b}$ is orthogonal to any plane determined by $\mathbf{a}$ and $\mathbf{b}$. (Bear in mind, the point $P$ was chosen arbitrarily. There are infinitely many choices for $P$, so there are infinitely many planes determined by $\mathbf{a}$ and $\mathbf{b}$, all parallel to each other.)

In the geometrical framework developed above, let $\ell$ be the line through $P$ and $R$, i.e., $\ell = \overleftrightarrow{PR}$. $\ell$ is perpendicular to $PAB$. Let $d$ be the distance between $P$ and $R$, i.e., $d = PR = |\mathbf{a} \times \mathbf{b}|$. There are exactly two points on $\ell$ whose distance from $P$ is $d$, one of which is $R$. Let $R'$ be the other point. $R$ and $R'$ lie on opposite sides of $PAB$. $\overrightarrow{PR}$ and $\overrightarrow{PR'}$ are directed line segments with equal length and opposite direction, so they represent opposite vectors; $\overrightarrow{PR}$ represents $\mathbf{a} \times \mathbf{b}$, and $\overrightarrow{PR'}$ represents $-(\mathbf{a} \times \mathbf{b})$. We now have two directed line segments originating at $P$, both perpendicular to $PAB$ and both having length $d$. If we see a picture of this situation where the tips are unlabeled, how would we know which is which? In other words, how would we know which directed line segment is $\overrightarrow{PR}$, representing $\mathbf{a} \times \mathbf{b}$, and which is $\overrightarrow{PR'}$, representing $-(\mathbf{a} \times \mathbf{b})$? What we need is a geometric principle that determines the *direction* of $\mathbf{a} \times \mathbf{b}$. We have such a principle; it is known as the **Right-Hand Rule**: If you curl the fingers of your *right* hand in the direction that $\mathbf{a}$ would rotate toward $\mathbf{b}$ through the angle $\theta$ between $\mathbf{a}$ and $\mathbf{b}$, then your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

So far, we have placed $\mathbf{a}$ and $\mathbf{b}$ both at $P$, so they are represented by $\overrightarrow{PA}$ and $\overrightarrow{PB}$, respectively. Let us now place $\mathbf{a}$ at tail $B$ and $\mathbf{b}$ at tail $A$. These new representations will have a common tip, which we name $Q$. We now have a parallelogram, $PAQB$, lying in plane $PAB$. Note that $\mathbf{a}$ is represented by both $\overrightarrow{PA}$ and $\overrightarrow{BQ}$, while $\mathbf{b}$ is represented by both $\overrightarrow{PB}$ and $\overrightarrow{AQ}$.

Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$, which is $\sphericalangle APB$ of parallelogram $PAQB$.

We shall consider side $\overline{PA}$ as the base of the parallelogram; its length is $a$. The length of side $\overline{PB}$ is $b$. Let $h$ be the height of the parallelogram. $\sin\theta = \frac{h}{b}$, so $h = b\sin\theta$.

For any parallelogram, its area is equal to the length of its base multiplied by its height. Thus, the area of $PAQB$ is $a$ times $b\sin\theta$, i.e., $ab\sin\theta$. By Theorem 2, this equals $|\mathbf{a} \times \mathbf{b}|$.

Furthermore, the area of $\Delta PAB$ is half the area of parallelogram $PAQB$, or $\frac{1}{2}ab\sin\theta$, and hence equals $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

We have thus proved the following theorem:

**Theorem 11**: If $\mathbf{a}$ and $\mathbf{b}$ are nonzero, non-parallel three-dimensional vectors, then they determine a parallelogram whose area is $|\mathbf{a} \times \mathbf{b}|$, and they determine a triangle whose area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

**Example 12**: In $x,y,z$ space, let $P = (3,5,2)$, $A = (5,6,5)$, $B = (4,8,3)$, and $Q = (6,9,6)$. $\overrightarrow{PA}$ and $\overrightarrow{BQ}$ represent the vector $\mathbf{a} = < 2,1,3 >$, while $\overrightarrow{PB}$ and $\overrightarrow{AQ}$ represent the vector $\mathbf{b} = < 1,3,1 >$. $\mathbf{a} \times \mathbf{b} = < -8,1,5 >$, whose magnitude is $\sqrt{90} = 3\sqrt{10}$. Thus, parallelogram $PAQB$ has area $3\sqrt{10}$ and $\Delta PAB$ has area $\frac{3}{2}\sqrt{10}$.

Note: If $\mathbf{a}$ and $\mathbf{b}$ are nonzero and non-parallel *two*-dimensional vectors, and we wish to find the area of the parallelogram they determine, we could use the formula $ab\sin\theta$, which is equally valid in both two and three dimensions. However, there is a more efficient approach. If $\mathbf{a} = < a_1,a_2 >$ and $\mathbf{b} = < b_1,b_2 >$, we can replace $\mathbf{a}$ with $< a_1,a_2,0 >$ and $\mathbf{b}$ with $< b_1,b_2,0 >$, and then compute $|\mathbf{a} \times \mathbf{b}|$. $< a_1,a_2,0 > \times < b_1,b_2,0 > = < 0,0,a_1b_2 - a_2b_1 >$, whose magnitude is $|a_1b_2 - a_2b_1|$. Thus, the area of the parallelogram determined by $< a_1,a_2 >$ and $< b_1,b_2 >$ is $|a_1b_2 - a_2b_1|$.

**Example 13**: In the $x,y$ plane, let $P = (1,1)$, $A = (3,3)$, $B = (4,7)$, and $Q = (6,9)$. $\overrightarrow{PA}$ and $\overrightarrow{BQ}$ represent the vector $\mathbf{a} = < 2,2 >$, while $\overrightarrow{PB}$ and $\overrightarrow{AQ}$ represent the vector $\mathbf{b} = < 3,6 >$. We can find the area of parallelogram $PAQB$ in either of the two following ways:

- $a = \sqrt{8} = 2\sqrt{2}$, $b = \sqrt{45} = 3\sqrt{5}$, and $\mathbf{a} \cdot \mathbf{b} = 18$, so $\cos\theta = \frac{18}{(2\sqrt{2})(3\sqrt{5})} = \frac{3}{\sqrt{10}}$.
  Therefore $\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{1}{\sqrt{10}}$, and the area of $PAQB$ is $(2\sqrt{2})(3\sqrt{5})\frac{1}{\sqrt{10}} = 6$.
- $|a_1 b_2 - a_2 b_1| = |12 - 6| = 6$.

Clearly, the second approach is far more efficient!

### 3. Additional Properties of the Cross Product:

**Theorem 12**: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. (You can use a circle diagram to remember this.)

**The Scalar Multiple Rule**: For any scalar $c$, $c(\mathbf{a} \times \mathbf{b}) = (c\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (c\mathbf{b})$.

If we take $c$ to be $-1$, and using the fact that $-\mathbf{v} = -1\mathbf{v}$, we get the following result:
$-(\mathbf{a} \times \mathbf{b}) = -\mathbf{a} \times \mathbf{b} = \mathbf{a} \times -\mathbf{b}$. In this equation, parentheses around $-\mathbf{a}$ and $-\mathbf{b}$ are implied.

The cross product is not commutative: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$.

**The Reversal Rule**: For any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$. By the same token, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. (When $\mathbf{a}$ and $\mathbf{b}$ are nonzero, if you switch the order of the vectors in the cross product, you get a vector with the same magnitude as before but in the opposite direction.)

The cross product is not associative: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

**The Double Cross Product Theorem**: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ Note that the result is a vector.

Convention: In an expression involving the cross product as well as vector addition or subtraction, the cross product is to be carried out first, unless parentheses are included to indicate otherwise. (Thus, the relationship between the cross product and vector addition or subtraction is the same as the relationship between ordinary multiplication and addition or subtraction.)

**Example 14**: In the expressions $\mathbf{a} \times \mathbf{b} + \mathbf{c}$ or $\mathbf{a} - \mathbf{b} \times \mathbf{c}$, the cross product would be carried out first, but in the expressions $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ or $(\mathbf{a} - \mathbf{b}) \times \mathbf{c}$, the addition or subtraction would be carried out first.

The **Distributive Property**:
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{c}$

In other words, the cross product distributes over vector addition and subtraction. Note that the result is a vector.

On the basis of the Distributive Property, the cross product of vector binomials can be distributed out, just like products of ordinary binomials (a process commonly known as FOIL).

**Example 15**: $(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} - \mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{d}$

Theorem 6 dealt with the expression $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. This calculation, combining the cross product with the dot product, is a very important one, as you will see shortly. The following theorem gives us some options for manipulating the expression...

**The Cross Product/Dot Product Theorem**:
$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
Note that the result is a scalar.

## 4. The Box Product or Triple Scalar Product:

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the **box product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Because it produces a scalar, it is also known as the **triple scalar product** of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This scalar can be positive or negative or zero.

In arithmetic, when we talk about "the quotient of two numbers," the *order* in which we state the numbers is important. For instance, the quotient of $12$ and $3$ is $4$, whereas the quotient of $3$ and $12$ is $0.25$. Likewise, the order in which we state the three vectors in a box product is important. For instance, the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, whereas the box product of $\mathbf{b}$, $\mathbf{a}$, and $\mathbf{c}$ is $(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$, which equals the *negative* of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

Since the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, the box product is zero if and only if $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{c}$. On the basis of this equivalence, several of our earlier theorems can be reformulated in terms of the box product...

**Theorem 13**: $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is zero.

Theorem 13 is a restatement of Theorem 9.

**Theorem 14**: If $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, then the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is zero and the three vectors are coplanar.

Theorem 14 is a restatement of Corollary 2 to Theorem 1 and Theorem 6.

**Theorem 15**: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero, non-parallel vectors. The following three conditions are equivalent (i.e., each one implies the other two):

1. The box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is zero.
2. $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.
3. $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Theorem 15 is a restatement of Theorem 8.

On the basis of the above theorems, if any of the three vectors is zero, or if any two of them are parallel, then the box product is zero. If the three vectors are nonzero and pairwise non-parallel, the box product may or may not be zero. If the box product is nonzero, then the three vectors must be nonzero and pairwise non-parallel.

**Example 16**: Let $\mathbf{a} = < 3,5,-4 >$, $\mathbf{b} = < -7,-2,6 >$, and $\mathbf{c} = < 6,10,-8 >$. Since $\mathbf{c} = 2\mathbf{a}$, the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $0$.

**Example 17**: Let $\mathbf{a} = < 4,1,3 >$, $\mathbf{b} = < 3,4,5 >$, and $\mathbf{c} = < 6,-5,-1 >$. Note that $3\mathbf{a} - 2\mathbf{b} = \mathbf{c}$. Since $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $0$.

The Cross Product/Dot Product Theorem gives us six equivalent ways of formulating the box product. However, *none* of these is an efficient way to compute the box product. The *efficient* way of computing the box product is to use the following theorem...

**The Box Product Theorem**: Let $\mathbf{a} = < a_1,a_2,a_3 >$, $\mathbf{b} = < b_1,b_2,b_3 >$, and $\mathbf{c} = < c_1,c_2,c_3 >$. Then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, also denoted $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

**Example 18**: Let $\mathbf{a} = < 1,2,-1 >$, $\mathbf{b} = < -2,0,3 >$, and $\mathbf{c} = < 0,7,-4 >$. The box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $\begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$. By the Cofactor Expansion along the third row, we get $-7\begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = -7(3-2) - 4(0+4) = -7 - 16 = -23$. (Since the box product is nonzero, the vectors must be non-coplanar.)

**5. Geometric Interpretation of the Box Product**:

**Theorem 16**: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-coplanar, then they determine a parallelepiped whose volume is equal to the absolute value of their box product–i.e., the volume is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

Note: The reason we must take the absolute value is that the box product itself could be a negative number. We always want volume to be positive, so we must take the absolute value of the box product ensure we get a positive answer.

In case you are not familiar with it, a **parallelepiped** is the three-dimensional version of a parallelogram, just as a rectangular box is the three-dimensional version of a rectangle and a cube is the three-dimensional version of a square. (A square is a special case of a rectangle, which is in turn a special case of a parallelogram. Similarly, a cube is a special case of a rectangular box, which is in turn a special case of a parallelepiped. If a parallelepiped has adjacent edges that are perpendicular, then it is a rectangular box; if, in addition, all its edges have equal length, then it is a cube.)

**Proof**:

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be non-coplanar vectors. (Hence, they are nonzero and pairwise non-parallel.) Place all three at a common tail $P$. Let $A$ be the tip of $\mathbf{a}$, let $B$ be the tip of $\mathbf{b}$, and let $C$ be the tip of $\mathbf{c}$, so $\overrightarrow{PA}$ represents $\mathbf{a}$, $\overrightarrow{PB}$ represents $\mathbf{b}$, and $\overrightarrow{PC}$ represents $\mathbf{c}$. $\overrightarrow{PA}$ and $\overrightarrow{PB}$ determine plane $PAB$. $\overrightarrow{PC}$ does not lie in $PAB$ (its tail is its only point in $PAB$).

$\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ determine a parallelepiped having adjacent edges are $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$. Let us name this parallelepiped $PABC$.

Place $\mathbf{a}$ at tail $B$ and $\mathbf{b}$ at tail $A$. These new representations will have a common tip, which we name $Q$. We now have a parallelogram, $PAQB$, lying in plane $PAB$. Note that $\mathbf{a}$ is represented by both $\overrightarrow{PA}$ and $\overrightarrow{BQ}$, while $\mathbf{b}$ is represented by both $\overrightarrow{PB}$ and $\overrightarrow{AQ}$.

$Q$ is one of the eight vertices of $PABC$, and $PAQB$ is the base of $PABC$.

By Theorem 11, the area of $PAQB$ is $|\mathbf{a} \times \mathbf{b}|$.

Let $h$ be the *vertical height* of $PABC$ (i.e., the height perpendicular to the base). The volume of any parallelepiped is the product of its vertical height and its base area; thus, the volume of $PABC$ is $h|\mathbf{a} \times \mathbf{b}|$.

(It is possible that $\mathbf{c}$ might be orthogonal to $PAB$, in which case its length, $c$, would be the vertical height $h$. However, this is not generally the case. If it is not, then we would refer to $c$ as the *slant height* of $PABC$, which is *not* what we need for calculating the volume.)

$\mathbf{a} \times \mathbf{b}$ is nonzero. Let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$, i.e., $\mathbf{u} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$. Like $\mathbf{a} \times \mathbf{b}$, $\mathbf{u}$ is orthogonal to $PAB$.

$h$ is the length of the vector projection of $\mathbf{c}$ onto $\mathbf{u}$. In other words, $h$ is the absolute value of the component of $\mathbf{c}$ along $\mathbf{u}$, i.e., $h = |comp_{\mathbf{u}}\mathbf{c}|$.

Since $\mathbf{u}$ is a unit vector, $comp_{\mathbf{u}}\mathbf{c} = \mathbf{u} \cdot \mathbf{c} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} \cdot \mathbf{c} = \frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.
So $h = |comp_{\mathbf{u}}\mathbf{c}| = \left|\frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right| = \frac{1}{|\mathbf{a} \times \mathbf{b}|}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

Thus, the volume of $PABC$ is $h|\mathbf{a} \times \mathbf{b}| = \frac{1}{|\mathbf{a} \times \mathbf{b}|}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}||\mathbf{a} \times \mathbf{b}| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

**QED**.

**Example 19**: Let $\mathbf{a} = < 1,2,-1 >$, $\mathbf{b} = < -2,0,3 >$, and $\mathbf{c} = < 0,7,-4 >$. We saw in Example 18 that the box product of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $-23$. Hence, the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $23$.